1 Research

This report details my research progress through the academic year 2011-2012 while supported by the Jefferson Science Associates Graduate Fellowship Program. My work during this time was split between two projects. These two projects are described below.

2 Static meson potentials and tetraquark bound states

A calculation of the interaction potential of a heavy-light heavy-light (HLHL) system in lattice QCD is used to study the existence of tetraquark bound states. The interaction potential of the tetraquark system is calculated on the lattice with 2+1 flavours of dynamical fermions with lattice interpolating fields constructed using colorwave propagators. The use of these propagators provides a new method of spatially smearing the interpolating fields, a technique which allows for a better overlap with the ground state wavefunction. Lattice HLHL potentials are extracted for 24 distinct channels, and are fit with a phenomenological non-relativistic quark model potential, from which a determination of the existence of bound states is made via numerical solution of the two body radial Schrodinger equation.

This work has culminated in a paper that will be submitted to Physical Review Letters D in the coming month. I’ve included a copy of the current draft of this work.

3 Charmed bottom baryon spectroscopy

The arena of doubly and triply heavy baryons remains experimentally unexplored to a large extent. This has led to a great deal of theoretical effort
being put forth in the calculation of mass spectra in this sector. Although
the detection of such heavy particle states may lie beyond the reach of ex-
periments for some time, it is interesting to compare results between lattice
QCD computations and continuum theoretical models. Several recent lattice
QCD calculations exist for both doubly and triply charmed as well as dou-
bly and triply bottom baryons. In this work we present preliminary results
from the first lattice calculation of the mass spectrum of doubly and triply
heavy baryons including both charm and bottom quarks. The wide range
of quark masses in these systems require that the various flavors of quarks
be treated with different lattice actions. We use domain wall fermions for
2+1 flavors (up down and strange) of sea and valence quarks, a relativistic
heavy quark action for the charm quarks, and non-relativistic QCD for the
heavier bottom quarks. The calculation of the ground state spectrum was
studied and compared to recent models.

I presented preliminary results from this work in a talk given at at the
Sixth International Conference devoted to Quarks and Nuclear Physics on
April 18th 2012 in Palaiseau, France. The published proceedings for this
talk can be found at: http://pos.sissa.it/archive/conferences/157/

Additionally, results were presented at the Jefferson 2012 Users Group
Meeting Poster session. This juried poster session offers awards for the top
three posters based on a juried evaluation of a short poster presentation,
and my poster tied for first place with another graduate student working at
Jefferson Lab.

This work on charmed bottom spectroscopy is still in progress, and we are
currently working to include another lattice volume and three more valence
quark masses.

4 Travel

Travel funding provided by the JSA fellowship was used to facilitate travel to
the Sixth International Conference devoted to Quarks and Nuclear Physics.
There I presented a talk entitled “Charmed bottom baryon spectroscopy.”
A calculation of the interaction potential of a heavy-light heavy-light (HLHL) system in lattice QCD is used to study the existence of tetraquark bound states. The interaction potential of the tetraquark system is calculated on the lattice with 2+1 flavours of dynamical fermions with lattice interpolating fields constructed using colorwave propagators. The use of these propagators provides a new method of spatially smearing the interpolating fields, a technique which allows for a better overlap with the ground state wavefunction. Lattice HLHL potentials are extracted for 24 distinct channels, and are fit with a phenomenological non-relativistic quark model potential, from which a determination of the existence of bound states is made via numerical solution of the two body radial Schrodinger equation.

I. INTRODUCTION

The calculation of hadronic forces from first principles allows insight into how interactions of the fundamental quark and gluonic degrees of freedom manifest themselves at the hadronic level. Lattice QCD is an excellent tool for calculating hadronic observables in the low energy regime. Although lattice calculations in euclidean space are not well suited for the study of real-time scattering processes, two methods can be used to extract interaction information from the lattice. One method, developed by Lüscher [1], relates the elastic scattering phase shift of a two particle system in a finite periodic box with the energy levels of the system. An alternate method, used in the present work, extracts the interaction energy as a function of hadron separation. This method is only applicable for systems of hadrons containing more than one heavy quark which can be treated in the static approximation providing a definite spatial position for the hadrons.

Phenomenologically, HLHL systems have become interesting in the study of tetraquark bound states [2] [3] [4]. It has long been known that the binding of a $QQ\bar{q}\bar{q}$ (with $q = u, d$) system
increases with the mass ratio of the heavy to light quark flavours [5], thus $c\bar{c}q\bar{q}$ and $b\bar{b}q\bar{q}$ systems are excellent candidates in the search for exotic four quark bound states. In Ref. [4] a distinction is made between two types of tetraquark bound states: molecular, in which the four quarks exhibit a single physical two-meson (singlet-singlet) component, and the more exotic compact bound states. The latter would involve a complicated color space structure in which quark pairs form color vectors which then combine to form a colorless four quark state [4]. In spite of this complicated color structure, compact bound states can be interpreted as a mixture of various two meson (color singlet) components [6]. The expected features that would characterize a molecular bound state would be a small binding energy and a bound state RMS radius roughly the same size as that of the sum of the two particle sizes, i.e.:

$$\Delta R \equiv \frac{RMS_{4q}}{RMS_{M_1} + RMS_{M_2}} > 1$$

A compact state, on the other hand, would be more tightly bound and have a smaller RMS radius than the molecular state. It is with these ideas in mind that we may begin to search for the signature of compact bound states on the lattice.

A recent lattice calculation of the HLHL interaction energy [7] has in fact hinted at the possibility of a bound tetraquark state in one channel that exhibits a particularly wide and deep potential well when compared with other channels, although no exhaustive determination of a bound state was undertaken. Our work goes beyond this presenting the first quantitative determination of a bound state energy in the HLHL system from a lattice calculation.

An inherent difficulty in making comparisons between theoretical models and lattice calculations performed in the static limit stems from the omission of the heavy quark spin in the static limit. As $m_H \to \infty$, the integer valued ($J = 0, 1$) angular momentum eigenstates of a single heavy light meson map onto a single static limit eigenstate with $J = 1/2$. The energies of the non-static angular momentum eigenstates also converge to a single energy corresponding to the $J = 1/2$ eigenstate. Although the two spaces map onto each other, there is not a simple one to one correspondence between static limit eigenstates with their non-static counterparts, and care must be taken in making identifications between the two spaces. Early lattice studies of HLHL interaction energy ([8] for example) differentiated states based on a subset of the available quantum numbers, while recently a more complete set of quantum numbers has been used exploiting the full set of symmetries of the HLHL system. [9] Our choice of quantum numbers (presented in section II) allows us to draw a connection between the quantum numbers and the qualitative behavior of the states. Additionally, by way of symmetry arguments, we are able to relate our static-limit states to non-
static angular momentum eigenstates.

II. BACKGROUND

A. Heavy-Light states

The quark model view of a heavy-light meson is of a heavy anti-quark $\bar{Q}$ coupled to a light quark $q$. The relevant quantum numbers to describe such a state are total angular momentum $J$ and its projection along some axis (here arbitrarily chosen to be $z$) $J_z$, and the parity $P_i$ as well as the relevant flavor quantum numbers. For our interests, we choose $\bar{Q} = \bar{b}$ and $q = \{u, d\}$. Therefore all states then have bottomness $b = -1$, and are otherwise classified by total isospin and the third component of isospin $(I, I^z) = (1/2, \pm 1/2)$ for $q = \{u, d\}$. Throughout this work, we make the assumption that we fit our correlation functions with a sufficiently large $t_{\text{min}}$ such that contributions from excited states have died out and we extract only the ground state energy. Throughout this work, we shall assume that states with non-zero orbital angular momentum $L$ are at sufficiently high energies as to have a negligible contribution to the ground state energies which we extract. We are then free to speak of the spin and total angular momentum interchangeably.

In heavy quark effective theory, spin dependent contributions enter into the heavy quark action at order $1/m_H$, and in the static limit ($m_H \to \infty$) the heavy quark acts as a static color source. This means that the spin of the HL meson comes only from the light degrees of freedom. Because of this the physical HL meson states with $J = (0, 1)$ become degenerate in the static limit, with both represented by a single $J = 1/2$ state. The relevant angular momentum classification is then $(J, J_z) = (1/2, \pm 1/2)$. With the above flavor assignments, the lowest energy excitations of the B spectrum with $J^P = \{0, 1\}^-$ (coupling to the static $J^P = 1/2^-$ B) are $B_{0, \pm}$ and $B^*$, and for $J^P = \{0, 1\}^+$ (coupling to the static $J^P = 1/2^+ B_1$), the ground state $B_1 (5721)^0$ (neglecting excited states).

B. Heavy-Light Heavy-Light states

When constructing states with a pair of HL mesons, care must be taken in determining a relevant set of quantum numbers that fully exploit the symmetries of the problem. The flavor quantum numbers for a Heavy-Light Heavy-Light (HLHL) system are straightforward, and for a $\bar{Q}q\bar{Q}q$ there are two isospin combinations, an isospin triplet with $I = 1$ and an $I = 0$ singlet. For a HLHL pair separated by a vector $\vec{r}$ the rotational symmetry is broken to rotations around the
separation axis. Total angular momentum $J$ is therefore no longer a conserved quantity, though its projection along the axis of separation (arbitrarily taken to be $\hat{z}$) is still conserved. The system will also be symmetric or antisymmetric under parity and reflections through a plane containing the separation axis, which we shall call $P_\perp$. This last transformation can be accomplished by a parity transformation followed by a rotation of $\pi$ about an axis perpendicular to the reflection plane. States with $J_z = \pm 1$ are not invariant under this transformation (being mapped onto each other), but their average is an eigenstate of $P_\perp$. Lastly we choose to classify HLHL states by intrinsic parity, defined to be the product of the intrinsic parities of the two light quarks, and interchange parity, defined as the product of the intrinsic parity transformation and coordinate inversion of the two particle spatial wavefunction.

We will use both the intrinsic and exchange parities in our classification of states.

### III. METHODOLOGY

#### A. HL and HLHL interpolating fields

A general interpolating operator coupling to a single heavy-light state is given by:

$$O_{HL}(x) = \bar{Q}(\vec{x}) \Gamma q(\vec{x})$$

with $\Gamma$ chosen to achieve the desired angular momentum and parity quantum numbers. For pseudoscalar HL states, $\Gamma = \gamma_5, \gamma_i$ (with $i = 1, 2, 3$), corresponding to a particle in the static limit with $J^P = 1/2^-$, which we will refer to simply as $B$. $J = 1$ meson states with $\Gamma = 1, \gamma_i \gamma_5$ correspond to a state with $J^P = 1/2^+$, which we shall refer to as $B_1$. We make the choice $\Gamma = \gamma_5$ for $O_B$ and $\Gamma = 1$ for $O_{B_1}$. As it will be useful in the analysis of HLHL states, it should be noted that for these choices of $\Gamma$, correlation functions constructed from $O_B$ interpolating fields will consist of only upper (positive parity) components in the Dirac basis of the light quarks while those constructed from $O_{B_1}$ will consist of only lower (negative parity) components. This is explicitly shown in Appendix A. The states are classified by the additional flavor quantum numbers $(I, I_z) = (1, \pm 1)$ for $q = \{u, d\}$.

For HLHL states, we want to create states with definite $(I, I_z, |J_z|) P_\perp P P_\perp$ and displacement $\vec{r}$ at the source and sink. To do this, we want to couple only our light quarks in spinor space to specify the quantum numbers of the state while allowing the heavy quarks to act only as color.
sources. Our general HLHL operator is then given by:

\[
\mathcal{O}^{(I,I_z,|J_z|,R\perp,P,P_i)}_{\text{HLHL}}(\vec{x},\vec{r}) = \bar{Q}(\vec{x},t) \bar{Q}(\vec{x} + \vec{r},t) \times [q(\vec{x},t) q(\vec{x} + \vec{r},t)]
\]

where the light quark wavefunctions \([q(\vec{x},t) q(\vec{x} + \vec{r},t)]\) are combined in such a way as to achieve the set of quantum numbers \((I, I_z, |J_z|, R\perp, P, P_i)\) of the system. The explicit construction of these wavefunctions is described in Appendix B. For simplicity we restrict ourselves to identical source and sink interpolating fields neglecting any cross correlators between states. Isospin is a good quantum number on the 2 + 1 flavor lattices with which we work, and we choose our interpolating fields to be isospin eigenstates with \((I, I_z) = (1, 1)\) and \((I, I_z) = (0, 0)\). At large spatial separations, we expect the energy of the four quark state to asymptotically approach the energy of its dominant two meson component\(^1\). States with \(P_i = -1\) will tend towards the energy of a \(BB_1\) combination at large spatial separations. There are two possible combinations of the light quark parities that yield \(P_i = +1\): \((p_1, p_2) = (+, +), (-, -)\). In light of the above discussion of parity content of single HL states, we project our \(P_i = +1\) interpolating fields to contain only negative or positive parity spinor components and retain these as distinct interpolating fields. The expectation is that interpolating fields constructed from lower spinor components will exhibit a significantly higher ground state energy in relation to those constructed from upper components. The reason for this is that the lower lower combinations are constructed from the same components as \(B_1\), and should exhibit an asymptotic energy (as \(\vec{r} \to \infty\)) near twice that of the single \(B_1\) energy, and similarly for upper components and \(B\) combinations. We differentiate all interpolating fields by their dominant asymptotic content in the tabulation of interpolating fields in Table I.

### IV. DETAILS OF THE LATTICE CALCULATION

We work with colorwave propagators (described below) calculated on \(n_f = 2 + 1\) anisotropic \((24^3 \times 128)\) lattices generated by the Hadron Spectrum Collaboration [11] with a pion mass of roughly 380 MeV. The fermion action used was the clover Wilson action with stout link smearing, not smeared in the temporal direction. The gauge action was Symanzik tree level tadpole-improved without a rectangle in the temporal direction, preserving temporal ultra-locality. The spatial and temporal lattice spacings for these lattices are \(a_s = 0.1227(8)\) fm and \(a_t = 0.03506(23)\) fm. The pion mass on this ensemble is 0.0681(4) in temporal lattice units. The Chroma Software package for

\(^1\) Here we are referring to the dominant lowest energy contribution, as we expect excited states to contribute negligibly to the extracted HLHL ground state energies.
Lattice QCD [12] was used to generate both colorwave and heavy propagators. The calculation of the HL and HLHL energies was performed using 305 gauge field configurations with eight sources spaced evenly in the temporal direction. Energies ground state energies were extracted using single exponential fits, with an appropriate $t_{\text{min}}$ determined from an effective mass plot.

### A. Colorwave Formalism

#### 1. Two quark states

Consider a general operator for a two quark mesonic state:

$$\mathcal{O}(\vec{x}) = \bar{q}_1(\vec{x}) \Gamma q_2(\vec{x})$$

where we assume for simplicity that the two quarks have different flavors. We seek to calculate the correlation function with localized interpolating fields: (averaged over spatial source and sink locations to increase statistics)

$$C(t, t_0) = \sum_{x,y} \left\langle \mathcal{O}(y) \mathcal{O}^\dagger(x) \right\rangle = \sum_x \sum_y \text{tr} \left( S_1(x, t_0|y, t) \Gamma S_2(y, t|x, t_0) \Gamma \right)$$
Following the methodology presented in [10], we now consider any complete set of orthonormal states \( \{ \phi_i(x) \} \) which satisfy:

\[
\sum_i \phi_i^*(x) \phi_i(y) = \delta(x - y), \quad \sum_x \phi_i^*(x) \phi_j(x) = \delta_{ij}
\]

and insert them twice into the above two point function:

\[
C_B(t, t_0) = \sum_{x,x'} \sum_{y,y'} \langle S_1(x, t_0 | y', t) \delta(y - y') \Gamma S_2(y, t | x', t_0) \delta(x - x') \rangle
\]

\[
= \sum_{x,x'} \sum_{y,y'} \left( S_1(x, t_0 | y', t) \sum_i \phi_i^*(y) \phi_i(y') \Gamma S_2(y, t | x', t_0) \sum_j \phi_j^*(x) \phi_j(x') \right)
\]

\[
= \sum_{i,j} S_{i,j}^1(t_0, t) \Gamma S_{i,j}^2(t, t_0) \Gamma
\]

where we have defined:

\[
S_{i,j}^1(t, t_0) \equiv \sum_{x,y} \phi_i^*(y) S(y, t | x, t_0) \phi_j(x)
\]

A convenient choice for the \( \{ \phi_i(x) \} \) is a plane wave basis\(^2\): \( \phi_i(x) \equiv \phi_p(x) = e^{-ipx} \delta_{s,s'} \delta_{c,c'} \). The delta functions here operate on color and spin. With this choice of basis, we define \( S_{i,j}^1(t, t_0) \equiv S^{p,p'}(t, t_0) \) to be colorwave propagators. The use of these propagators allows us to implement spatial smearing at the source and sink of our correlation functions. In the limit where all momenta are summer over in equation 1, all to all point point propagators are recovered. As as we reduce the maximum momentum cutoff \( p_{\text{cut}}^2 \) we increase the effective amount of spatial smearing. The effect of restricting the plane wave basis to \( |p|^2 \leq |p_{\text{cut}}|^2 \) (summing over a momentum space volume) is illustrated in Fig. 1 where effective masses for single HL B Correlation functions is presented. It’s evident that the noise of the signal decreases by increasing the momentum space cutoff (as this increases the statistics contributing to the correlation function). Each effective mass plateau appears to begin at roughly the same point independent of \( |p_{\text{cut}}|^2 \), indicating that the low \( |p_{\text{cut}}|^2 \) contributions couple more strongly to the ground state than the high \( |p_{\text{cut}}|^2 \) contributions do to excited states. In Fig. 2 we present the effective mass plots for correlators constructed from colorwave propagators with the restriction \( |p|^2 = |p_{\text{cut}}|^2 \) (summing over spherical shells in momentum space). These signals have significantly more noise due to lower stitistics from the momentum space restriction (summing over a spherical surface vs. a spherical volume in momentum space). It is

\(^2\)It should be noted that this basis is only valid in the Coulomb gauge. Although covariant smearing methods are commonly used today (to avoid costly gauge fixing), early lattice calculations often implemented Gaussian spatial smearing in the Coulomb gauge.
clear however that correlation functions constructed with only from colorwaves with larger values of $|p_{cut}^2|$ couple more to excited states as indicated by the slower convergence to the extracted ground state energy for increasing $|p_{cut}^2|$.

Because the Coulomb gauge is a smooth gauge, the low momentum degrees of freedom correspond directly to low energy excitations of the spectrum. This allows us to couple to different excitations of the spectrum by summing over various subsets of the momentum space.

2. HLHL States

We begin with a correlation function for two heavy-light mesons separated by $\vec{r}$ as described above:

$$C_{HLHL}(t, \vec{r}) = \sum_x \langle \mathcal{O}_{HLHL}(\vec{x}, \vec{r}, t) \mathcal{O}_{HLHL}^\dagger(\vec{x}, \vec{r}, 0) \rangle$$

$$= \sum_x \langle \bar{Q}(\vec{x}, t) \bar{Q}(\vec{x} + \vec{r}, t) q(\vec{x} + \vec{r}, t) q(\vec{x} + \vec{r}, 0) q(\vec{x}, 0) Q(\vec{x} + \vec{r}, t) Q(\vec{x}, t) \rangle$$

Each heavy quark source can only be contracted with the sink at the same spatial location, and upon contraction we work only with the Wilson line portion of the heavy quark propagator, as we want the quantum numbers of the system to be determined entirely by the light degrees of freedom. There are two possible light quark contractions, one where the light quarks contract with source and sink at the same spatial location (direct), and one where the light quarks contract at the other spatial location (crossed). Performing these contractions, we have (omitting the overall color trace):

$$C_{HLHL}(t, \vec{r}) = \sum_x W^\dagger(\vec{x}; t_0, t) W^\dagger(\vec{x} + \vec{r}; t_0, t)$$

$$\times tr_d \left[ S(\vec{x} + \vec{r}, t; \vec{x} + \vec{r}, t_0) - S(\vec{x} + \vec{r}, t; \vec{x}, t_0) \right]$$

Here, $tr_d$ denotes the trace over Dirac space spinor indices.

We now introduce our partially fourier transformed light quark propagators as:

$$S(x'; t; x, t_0) = \sum_{p_1, p_1'} e^{i p_1 x'} S(p_1'; t; p_1, t_0) e^{-i p_1 x}$$

and the above correlator can be rewritten as:

$$C_{BB}(t, \vec{r}) = \sum_{p_1, p_1'} \sum_{p_2} W(\vec{x}; t_0, t) W(\vec{x} + \vec{r}; t_0, t)$$

$$\times e^{i(p_1' - p_1 - p_2 - p_2) x} e^{i(p_1' - p_2) r}$$

$$\times \left[ S(p_2'; t; p_2, t_0) S(p_1'; t; p_1, t_0) - S(p_2'; t; p_1, t_0) S(p_1'; t; p_2, t_0) \right]$$
FIG. 1: Effective mass for HL $B$ for increasing $|p^2| \leq |p^2_{\text{cut}}|$
FIG. 2: Effective mass for HL \( B \) for increasing \(|p^2| = |p_{\text{cut}}^2|\)
Defining

\[ D(\vec{r}, t, t_0, \omega) \equiv \sum_x G_H(\vec{x}; t_0, t) G_H(\vec{x} + \vec{r}; t_0, t) e^{i(\omega) x} \]

with \( \omega \equiv p'_1 - p_1 + p'_2 - p_2 \), our the final form of our HLHL correlation function becomes:

\[ C_{BB}(t, \vec{r}) = \sum_{p_1, p'^1_2, p'_1, p_2} D(\vec{r}, t, t_0, \omega) \times e^{i(p'_2 - p_2)r} \times [S(p'_2, t; p_2, t_0) S(p'_1, t; p_1, t_0) - S(p'_2, t; p_1, t_0) S(p'_1, t; p_2, t_0)] \]

With this method, we calculate the costly \( D(\vec{r}, t, t_0, \omega) \) first and then perform the far less expensive contractions with the colorwave propagators for our complete operator basis.

V. HLHL RESULTS

For \( q = \{u, d\} \) we have 24 unique HLHL corresponding to the operators enumerated in Table I. Each potential curve is calculated by taking the jackknife difference between the energy of the HLHL state for various \( \vec{r} \) and the energy of the expected two meson asymptotic state:

\[ V(\vec{r}) = E_{HLHL}(\vec{r}) - E_{B(1)} - E_{B(1)} \]

The statistical uncertainty for each point is determined from jackknife statistical analysis. The systematic uncertainty is determined by adjusting the chosen fit range by one time slice in each direction and averaging the observed deviations in the energy. The systematic uncertainty for both \( E_{HLHL} \) and \( E_{B(1)} \) are determined independently and then added in quadrature to determine the systematic uncertainty on \( V(\vec{r}) \).

We find three different asymptotic values for the various states as illustrated in Fig. [3] (here which we take the energy difference for the three potential curves with respect to the \( BB \) energy). The lowest lying asymptotic value corresponds to states with a positive intrinsic parity \( P_i \) with all spin components in the correlation function projected to the upper spin components, while the highest asymptotic value corresponds to states with positive intrinsic parity and all spins projected to the lower components. This asymptotic behavior is in line with our expectation that the spin projection of our positive intrinsic parity operators increases the coupling to the different two meson components of the HLHL state. The energy difference between the two states is roughly twice the energy difference between the single HL \( B \) and \( B_1 \) states, indicating that they are both tending asymptotically towards their expected two meson asymptotic energies at long distances.
FIG. 3: Comparison of two higher asymptotic states to lowest asymptotic state

It is clear that there is some contamination from mixing of the HL $B_1$ with a $\pi - B$ state which is most evident in the overshoot of the highest asymptotic state beyond it’s expected value of twice the $B_1$ energy for $d > 0.8$ fm. All $P_1 = (-)$ states exhibit an asymptotic tendency towards the sum of the single HL $B$ and $B_1$ energies as expected.

As they exhibit the cleanest signal with the least contamination from higher energy $\pi - B$ states, we will focus mainly on the $BB$ energies, which we present in Fig. [4]. Several aspects of these potential curves should be noted: First, we find that the product of exchange parity $P$ and intrinsic parity $P_1$, which is the symmetry of the system under particle exchange, determines the attractiveness ($-$) or repulsiveness ($+$) of the state. This is in agreement with [7]. Second, the $(I, I_z, |J_z) P_\perp, P, P_1 = (0, 0, 0) + - +$ exhibits a significantly deeper and wider potential well when compared with the two other attractive channels. This qualitative difference was acknowledged in [7], and the quantum numbers of this channel are consistent with a bound state predicted in a phenomenological model in [4].

VI. BOUND STATES

As the HLHL system has been predicted to be an excellent candidate for bound tetraquark states, we seek a quantitative method for extracting such a bound state (if one exists) from our lattice calculation. Our method is as follows: We fit our lattice potential to a phenomenological quark model potential as described in [13]. We make the choice to focus on the $(I, I_z, |J_z) P_\perp, P, P_1 = (0, 0, 0) + - +$ channel, as previous work has hinted at the possibility of a bound state here. As a control, we also perform the fit for the $(I, I_z, |J_z) P_\perp, P, P_1 = (1, 1, 0) - - +$
FIG. 4: Calculated HLHL $BB$ energies with expected asymptotic value (twice the calculated HL $B$ mass)
channel as well. In our fit, we neglect the $\vec{r} = 0$ points as the finite value of the potential at $\vec{r} = 0$ is a lattice artifact stemming from the ultraviolet cut off introduced by the lattice discretization, leaving us with 7 data points for each potential curve, and two free parameters from the fit model. The model with the extracted fit parameters is then taken to be the interaction potential between two B mesons in the continuum limit. The two body (one-dimensional) Schrodinger equation is then solved numerically with this interaction potential to determine the existence of any negative energy (bound) states. It should be noted here that the solutions to the Schrodinger equation will converge to their continuum values as the continuum limit of the lattice calculation is taken. As we have only a single lattice spacing available to work with this continuum extrapolation is not an option, and it should be understood that the results presented in this section are at finite lattice spacing.

A. Potential Model

We have limited our displacements $|\vec{r}| \leq 1.27$ fm, therefore long range effective interactions due to meson exchange do not provide a good description of the HLHL system. In reference [13], a quark model picture of a two meson interaction was used to derive an interaction potential for the HLHL system, which included color coulomb, spin-spin, linear confinement interactions. Details of the derivation of the potential model can be found in the aforementioned reference, and we will only highlight several modifications we make when fitting this potential model to our numerical results. The quark model HLHL potential has the form:

$$V_{BBDS}(r) = C_I V_{cc} (\alpha_s, \beta, r) + C_S s V_{ss} (\alpha_s, \beta, \bar{m}, r) + C_I V_{lc} (b, \beta, r)$$

with:

$$V_{cc} (\alpha_s, \beta, r) = \frac{-4 \alpha_s}{9r} \left[ 1 + \left( \frac{2}{\pi} \right)^{1/2} \frac{\beta r}{2} \text{Erf} \left( \frac{\beta r}{2} \right) \right] e^{-\beta^2 r^2/2}$$

$$V_{ss} (\alpha_s, \beta, \bar{m}, r) = \frac{2}{27} \left( \frac{2}{\pi} \right)^{1/2} \frac{\alpha_s \beta^3}{\bar{m}^2} e^{-\beta^2 r^2/2}$$

$$V_{lc} (b, \beta, r) = \frac{b}{3 \beta} \left[ \beta r e^{-\beta^2 r^2/2} + 2 \left( \frac{2}{\pi} \right)^{1/2} e^{-\beta^2 r^2/2} - \left( \frac{\beta r}{2} \right) \text{Erf} \left( \frac{\beta r}{2} \right) e^{-3 \beta^2 r^2/4} \right]$$

Here, $\alpha_s$ is the strong coupling constant, $\beta$ is the spatial width of the quark model single HL meson wavefunction, $\bar{m}$ is the mass of the light quark in the $\overline{MS}$ scheme, and $b$ is the QCD string
tension. The coefficients $C_I$ and $C_{S\cdot S}$, which contain the spin information of the HLHL state, are defined as matrix elements between initial (unprimed) and final (primed) two meson states and will be discussed further below. It should be noted that the above potential model acquires an overall minus sign if the isospin wavefunction of the two meson state is antisymmetric. Notice that the potential is a function of $|\vec{r}|$ and not $\vec{r}$, as any tensor interaction terms are neglected in this model.

### B. Fit Model

When applying the above model to our lattice data, we must make several modifications to the above quark model potential. Due to the use of periodic boundary conditions in the calculation, interactions with image charges lying past the boundary must be accounted for. The range of these interactions is much larger than the range of displacements used in the calculation ($\geq 3$ fm), therefore we must also consider the possibility that there will be long range meson exchange interactions that were neglected in our choice of potential model. To account for these long range interactions with the images, we extend the original model by adding a simple Yukawa like term for one pion exchange:

$$V_{Yuk}(r) = V_{BBDS}(r) + \frac{g e^{m_\pi r}}{r}$$

Here we take $m_\pi$ to be the mass of the pion on the gauge field configurations used in the calculation ($\sim 390$ GeV). The parameter $g$ is discussed below.

In principle, interactions with each of the infinitely many image charges contribute to the potential and must be included. In practice however, we may restrict ourselves to contributions where the image of the first meson is $\leq 3L/2$ ($\sim 4.5$ fm) away from the second and vice versa. This truncation is valid as the potential (with the choice of parameters outlined below) at separation $r = 3L/2$ is $O(10^{-4})$ MeV, a negligible contribution to the overall potential. With the inclusion of the image charges our potential model then becomes:

$$V_{Yuk}^{Im}(r) = V_{Yuk}(r) + 2 \sum_{r_i < 3L/2} V_{Yuk}(r_i)$$

The addition of these image charges modify the potential at long distance as illustrated in Fig. 5

The final modification made to the potential model is a modification of the spin dependent coefficients $C_I$ and $C_{S\cdot S}$. The original presentation of this phenomenological potential model in Ref. [13] sought to provide a comparison with the lattice calculations of the time, which had
an incomplete classification of the HLHL states in terms of the total isospin $I$ and spin $S$ of the system, while also maintaining a connection with the physical $B$ meson states. Because of this, classification of the various potentials was made in terms of the physical $B$ and $B^*$ (first angular excitation of the $B$ meson) with respect to the quantum numbers $I$ and $S$.

The difference in angular momentum spaces of the non-static and static limit prevents a direct interpretation of the lattice data from the present work in terms of physical $B$ and $B^*$ states,
and our classification of states makes it difficult to reconcile the previous classification with ours. We therefore choose to recalculate the spin dependent coefficients of the potential model to better suit our calculation, the results of which are presented in Table II (For details of the calculation, see Appendix C) The previous determination of these coefficients for the HLHL system included spin degrees of freedom for the heavy quarks in the two meson states $|M_i M_j>$ allowing for better classification of the potential in terms of non-static limit states. We choose to neglect the spin degrees of freedom of the heavy quarks in our determination, effectively fully implementing the static limit for the potential model. Thus the spin degrees of freedom of our two meson kets $|M_i M_j>$ are just those of the spin of the light degrees of freedom of our HLHL state. The evaluation of these coefficients however requires knowledge of the total angular momentum of the two meson state, a point that has been neglected until now. As we seek to fit the $(I, I_z, |J_z>) P_\perp, P, P_i = (0, 0, 0) + - -$ and $(1, 1, 0) - -$ states, we need to determine if these particular states are in a symmetric angular momentum triplet, or an antisymmetric angular momentum singlet. In order to make this identification, we must rely on the overall symmetries of the state in question. We know that the interchange parity $P$ of a given state is the product of the intrinsic parity $P_i$ and the symmetry of the spatial wavefunction. From this relationship, and with knowledge of the symmetry of the isospin spatial wavefunction, we can infer the symmetry of the angular momentum wavefunction:

$$Sym_J = (-) (Sym_I) (P_i) (P) ,$$

where $Sym_J$ and $Sym_I$ the symmetries of the angular momentum and isospin wavefunctions. The overall negative sign appears from exchanging fermions in the parity operation. Using this we are able to identify the $(I, I_z, |J_z>) P_\perp, P, P_i = (0, 0, 0) + - -$ channel with $Sym_J = -$ as a $J = 0$ state, and the $(I, I_z, |J_z>) P_\perp, P, P_i = (1, 1, 0) - -$ channel with $Sym_J = +$ as a $J = 1$ state. The spin dependent coefficients can then be recalculated for our states and are shown in Table [II].

C. Fitting Procedure and Bound State Determination

In fitting the above mode to our lattice data, we use two free fit parameters: $\beta$ and $g$ and take the remaining parameters $b$, $\bar{m}$ and $\alpha_s$ to be 0.18GeV$^2$, 0.33GeV, and 0.5 respectively as in Ref. [13]. A fit is performed for each of 305 jackknife ensembles, allowing for an accurate way to estimate the error on the extracted fit parameters, shown in Table [II]. As we are ultimately interested in the energy levels allowed by the potential model, and not the model parameters themselves, we will
only briefly comment on the fit parameters. It is immediately obvious that \( g \) is not well determined for the \( J = 1 \) channel. It’s also interesting that the fit parameter \( \beta \) is significantly smaller for the \( J = 0 \) channel, indicating a much narrower spatial distribution of the two meson wavefunction.

Once the fit parameters have been extracted they are then inserted into the two body radial Schrödinger equation to determine if any bound states exist. As we are restricting ourselves to \( L = 0 \) states, the two body Schrödinger equation can be written as:

\[
\left[-\frac{\hbar^2}{2\bar{m}} \frac{d^2}{dr^2} + V_{Yuk}(r)\right] u(r) = Eu(r)
\]

where \( \bar{m} \) is the reduced mass of a two B meson system (with the single meson mass taken from the Particle Data Group [14]), \( u(r) = r\Psi(r) \) and \( V_{Yuk}(r) \) is the potential model presented in the preceding section excluding the image terms.

Eqn. 3 is then solved numerically as an eigenvalue problem with a spatial discretization of 0.01 fm and a spatial cutoff of 10 fm (corresponding to a sphere with \( r = 10 \) fm), and the boundary condition that \( \Psi(r) \bigg|_{r=10} = 0 \). This spatial volume provides ample space for the potential to decay to zero. The eigenvalue spectrum is then analyzed for each of the two states discussed above. While the \( J = 1 \) channel exhibits a near continuum of positive eigenvalues (discrete only because of the numerical solution method), the \( J = 0 \) channel does admit a single bound state with energy \( E_0 = -50.0(5.1) \) MeV (with the uncertainty determined by carrying through the jackknife analysis from the fit parameters and solving Eqn. 3 for each of the 305 \((\beta, g)\) sets). Aside from the binding energy, we can also calculate the RMS radius for the two meson wavefunction \( \Psi(r) \) from the wavefunctions \( u(r) \) above:

\[
r_{\text{RMS}} \equiv \langle r^2 \rangle^{1/2} = \frac{\sum_i r_i^2 |u(r_i)|^2}{\sum_i |u(r_i)|^2}
\]

For the bound state wavefunction \( u_0(r) \), we find an RMS radius of 0.383(6) fm, the error again estimated by jackknife analysis.

Although no previous calculation of the binding energy in this particular static-limit channel exists (lattice or otherwise), Ref. [4] does quote binding energies and RMS radii for a doubly bottom \( J^P (L, S, I) = 0^+ (0, 0, 0) \) channel which is consistent (in the static limit) with the quantum numbers of our static limit \((I, I_z, |J_z\rangle) P\downarrow, P, P_i = (0, 0, 0) + - + \) channel. This reference uses two different potential models to calculate binding energies: the constituent quark cluster model CQC and the Bhaduri-Cohler-Nogami or BCN model. The BCN model includes the same interactions as those used in Ref. [13] to derive the potential used to fit our lattice results (namely, color
coulomb, linear confinement and spin-spin). Furthermore, the BCN parameters corresponding to string tension $b$, strong coupling $\alpha_s$, and constituent quark mass $\bar{m}$ used in [4] are very similar to those used in our potential model (compare our $(b, \alpha_s, \bar{m}) = (0.18 \text{ GeV}^2, 0.5, 0.33 \text{ GeV})$ to $(0.186 \text{ GeV}^2, 0.52, 0.337 \text{ GeV})$). These binding energies should provide a relevant point of comparison for our results provided our lattice discretization errors have minor effects on the extracted potential model fit parameters. In comparison, we find our values for the binding energy and RMS radius to be consistent with the values quoted in [4] from the BCN model $(E_0, r_{\text{RMS}}) = (-52 \text{MeV}, 0.334 \text{fm})$, providing a good cross check that our lattice calculation has identified a bound state in the static limit $(I, I_z, |J_z|) P_\perp, P, P_i = (0, 0, 0) + - +$ channel. The fact that the bound state identified in that work has an RMS radius that is smaller than the sum of the individual mesonic RMS radii is indicative of the existence of the compact nature of that bound state. Additionally, the decomposition of our static limit state into various single meson spin combinations (as illustrated in Ref. [9] eg.), indicates that our static limit HLHL state is composed of several single meson components which is consistent with the expected behavior of a compact tetraquark bound state. In spite of this evidence however, no definitive statement can be made at the present time as to the compactness of the bound state identified from our lattice calculation.

<table>
<thead>
<tr>
<th>$(J, J_z)$</th>
<th>$C_I$</th>
<th>$C_{S,S}$</th>
<th>$\beta \text{ (GeV)}$</th>
<th>$g$</th>
<th>$\chi^2/d.o.f.$</th>
<th>$E_0 \text{ (MeV)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>-1</td>
<td>3/4</td>
<td>0.274(14)</td>
<td>0.041(12)</td>
<td>0.9943</td>
<td>-50.0(5.1)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>1</td>
<td>1/4</td>
<td>0.459(38)</td>
<td>0.016(20)</td>
<td>0.4119</td>
<td>N/A</td>
</tr>
</tbody>
</table>

TABLE II: Recalculated spin dependent coefficients from reference [15] and fit parameters from fitting our lattice data to a modified version of the model presented in ref. [13]. Here $\beta$ corresponds to the spatial width of the two meson wavefunction, and $g$ is the coupling strength of the additional Yukawa term introduced in this work. The uncertainties quoted for the fit parameters are jackknife estimates.

VII. CONCLUSIONS

We have carried out the first lattice motivated determination of bound state energies in the heavy light heavy light tetraquark system. The use of colorwave propagators in calculating meson correlation functions was presented, and extended to the HLHL system in order to provide a novel way by which to divide the calculation of HLHL correlation functions for different $(I, I_z, |J_z|) P_\perp, P, P_i$ channels. The effect of limiting the colorwave plane wave basis on the ground state overlap of single HL correlation functions was explored, including adjusting the maximum plane wave basis.
cutoff $p^2_{\text{cut}}$ and summing over various subsets of the plane wave basis. It was found that lower values of $p^2_{\text{cut}}$ generally provided the fastest convergence to the extracted ground state of the HL state, while increasing $p^2_{\text{cut}}$ had varying effects for each of the summation methods.

HLHL potentials were calculated for 24 distinct $(I, I_z, |J_z|) P_\perp, P, P_i$ channels, and exhibited three distinct asymptotic values corresponding to the meson meson content of the HLHL state. Contamination was apparent in the higher asymptotic values, visible in the overshoot of the energy of the HLHL state from the expected value of the two meson energy. This contamination likely stems from mixing of the $B_1$ with a $B - \pi$ state. It was determined that the attractiveness or repulsiveness of the HLHL potential was determined solely by the symmetry of the system under particle exchange, in agreement with Ref. [7]. The asymptotic behavior of the various HLHL states was shown to be dependent on the intrinsic parity of the state. While the $P_i = -$ states have only one asymptotic value (corresponding to a single two meson $BB_1$ component), the $P_i = +$ channels have two asymptotic values corresponding to both $BB$ an $B_1B_1$ two meson components. Examining the construction of single HL correlation functions, it was determined that we could increase overlap with the $BB$ and $B_1B_1$ two meson wavefunctions by projecting the correlation functions to include only positive or negative parity components of the Dirac space quark spinors.

The existence of bound states was then explored for the $(I, I_z, |J_z|) P_\perp, P, P_i = (0, 0, 0) + --$ channel as it exhibited a wider and deeper potential when compared with the other attractive potentials. Analysis was also carried out for the $(I, I_z, |J_z|) P_\perp, P, P_i = (1, 1, 0) -- +$ for the purposes of comparison. A modified version of the potential model described in Ref. [13] was used to fit the lattice data, and two fit parameters $\beta$ (the gaussian width of the two meson wavefunction) and $g$ (the Yukawa interaction constant) were extracted from the fit. Inserting the potential with extracted fit parameters into the two body Schrodinger equation, we then solved numerically for the eigenvalues of the hamiltonian, searching for any negative energy eigenstates. A single negative energy bound state was found in the $(0, 0, 0) + --$ channel, with an energy of $E_0 = -50.0(5.1)$ MeV and RMS radius $r_{\text{RMS}} = 0.383(6)$ fm. These results were found to be consistent with results presented in Ref. [4] for the state $J^P (L, S, I) = 0^+ (0, 0, 0)$ (which maps onto our $(0, 0, 0) + --$ channel in the static limit).

Acknowledgments

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Appendix A: Parity content of HL interpolating operators

Here we show that correlation functions for our $B(B_1)$ states are composed entirely of upper(lower) components of the Dirac space components of the ldof. We begin with a general HL correlation function with arbitrary source and sink operators (neglecting color indices and working in the Dirac basis):

$$C_B(t)_{i,j} = \sum_{\vec{x}} \left\langle O_{B_i}(\vec{x},t)O_{B_j}^\dagger(\vec{x},0) \right\rangle$$

$$= \sum_{\vec{x}} \left\langle \bar{Q}(\vec{x},t)\Gamma_i q(\vec{x},t)\bar{q}(\vec{x},0)\Gamma_j Q(\vec{x},0) \right\rangle$$

$$= \sum_{\vec{x}} tr \left( \gamma_5 (S_H(\vec{x},t;0))^\dagger \gamma_5 \Gamma_i S_L(\vec{x},t;0) \Gamma_j \right)$$

$S_H$ is a heavy quark propagator given by:

$$S_H(\vec{x},t; t_0) = \left( \frac{1 + \gamma_4}{2} \right) W(\vec{x},t; t_0) = P_+ W(\vec{x},t; t_0)$$

where $W(\vec{x},t; t_0)$ is a Wilson line from $t_0$ to $t$. Substituting this, we have:

$$C_B(t) = \sum_{\vec{x}} tr \left( \gamma_5 (P_+ W(\vec{x},t;0))^\dagger \gamma_5 \Gamma_i S_L(\vec{x},t;0) \Gamma_j \right)$$

$$= \sum_{\vec{x}} tr_c \left( W(\vec{x},t;0) tr_d \left( \Gamma_i P_- \Gamma_j S_L(\vec{x},t;0) \right) \right)$$

where we have used $\gamma_5 P_+ \gamma_5 = P_-$. For $\Gamma_i = \Gamma_j = 1$, we project to only the lower components of the Dirac space light quark propagator, while for $\Gamma_i = \Gamma_j = \gamma_5$ we project only to the upper components of the Dirac propagator.

Appendix B: Construction of light quark wavefunctions

To determine two quark wavefunctions in spin and flavor space yielding the quantum numbers $(I, I_z, J_z, P_\perp, P, P_i)$, we begin with states of definite $(I, I_z, J, J_z, P_i)$:

$$[q_1(p_1) q_2(p_2)]_{(I,I_z,J,J_z,P_i)} = \sum_{m_1,m_2} W_{I_1,I_2}^{J_1,J_2} W_{t_1,t_2}^{I_1,I_2} q_1(m_1,t_1,p_1) q_2(m_2,t_2,p_2)$$
where \( m, t, p \) are the projections of spin and isospin along the z-axis and the intrinsic parities of the light quarks, and the \( W^{J,J_z}_{m_1,m_2} = \langle 1/2, m_1, 1/2, m_2 | J, J_z \rangle \), \( W^{I,I_z}_{t_1,t_2} = \langle 1/2, t_1, 1/2, t_2 | I, I_z \rangle \) are the Clebsch-Gordon for angular momentum and isospin. From these operators, we average over \( J_z = \pm 1 \) states and determine \( P_{\perp} \) from the quantum numbers \( P_i \) and \( P \) and the spatial symmetry of the operator. It should be noted here that there are two combinations of \((p_1, p_2)\) that contribute to the \( P_i = +1 \) HLHL states, and we make the decision to keep these as distinct operators.

Linear combinations of the above operators are then taken to produce states of definite exchange parity \( P \), the necessary combinations determined by summing over sets of the above operators that map onto each other under \( P \) with the appropriate weight \( W^{P}_{p_1,p_2} = \pm 1 \)

\[
[q_1 q_2]_{(I,J_z,J_z,R_{\perp},P,P_i)} = \sum_{p_1,p_2} W^{P}_{p_1,p_2} [q_1 (p_1) q_2 (p_2)]_{(I,J_z,J_z,R_{\perp},P,P_i)}
\]

### Appendix C: Determination of spin coefficients for potential model

Here we present our derivation of the spin coefficients \( C_I \) and \( C_{S\cdot S} \) presented in Table II. In Ref. [15], an interaction potential for two meson states is calculated by including spin-spin, color coulomb and linear confinement interactions in a two quark interaction hamiltonian. By considering these interactions between each of the quark quark pairs in a 4 quark (2 meson) scattering state, transfer matrix elements are calculated and then Fourier transformed to give a corresponding position space potential. In Ref. [13], this method was applied to the HLHL system. When calculating the spin dependent portion of the potential, all but one of the interaction diagrams (referred to as “Transfer 2”) can be neglected because the spin of the heavy quarks is neglected in the static approximation. This diagram includes an insertion of the interaction hamiltonian between the two light quarks, as illustrated in in Fig. 7. The spin dependent contribution of this diagram to the potential can be factorized such that all the dependence enters through two coefficients, which are defined as matrix elements between the initail and final two meson states:

\[
C_I = \langle CD | I | AB \rangle \\
C_{S\cdot S} = \langle C_i D_j | S_i \cdot S_j | A_i B_j \rangle
\]

Where \( I \) here is understood to be the identity operator in spinor space. Upon inspection of the diagram, it’s clear that the matrix element of \( I \) will not always trivially be unity due to the quark interchange between the initial and final two meson state.
With respect to Fig 7, these matrix elements as outlined in [15] are defined explicitly as:

\[ C_I = \langle CD | I | AB \rangle = \chi_\lambda^C s_c, s_d | s_a, s_b > \delta_{sb, a, s_c} \delta_{sb, b, s_d} \chi_\lambda^A s_a, s_d | s_b, s_b \]

\[ C_{SS} = \langle C_i D_j | S_i \cdot S_j | A_i B_j \rangle = \chi_\lambda^C s_c, s_d | s_a, s_d > \delta_{sb, a, s_c} \delta_{sb, b, s_d} \chi_\lambda^A s_a, s_d | s_b, s_b \]

For our purposes, we wish to entirely neglect the spin of the heavy quarks in the above matrix elements. Because of this, the Clebsch-Gordan coefficients \( \chi_\lambda^C s_c, s_d \) etc. (relating the spin of the two quark state to the meson state) are all unity. The states between which we wish to calculate these matrix elements are two particle angular momentum eigenstates \( | J, J_z \rangle_{a,b} \equiv | s_a, s_b \rangle_{J, J_z} \), of which we are only interested in \( |1, 0\rangle \) and \( |0, 0\rangle \). To account the light quark exchange in Fig. 7, we note the following relations:

\[ | s_a, s_b \rangle_{J=0, J_z=0} = \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle) = -\frac{1}{\sqrt{2}} (| \downarrow \uparrow \rangle - | \uparrow \downarrow \rangle) = -| s_c, s_d \rangle_{J=0, J_z=0} \]  

(C1)

and

\[ | s_a, s_b \rangle_{J=1, J_z=0} = \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle) = \frac{1}{\sqrt{2}} (| \downarrow \uparrow \rangle + | \uparrow \downarrow \rangle) = | s_c, s_d \rangle_{J=1, J_z=0} \]  

(C2)

From the above relations, it is easy to calculate the matrix elements of interest for our problem (for the states \( |1, 0\rangle \rightarrow |1, 0\rangle \) and \( |0, 0\rangle \rightarrow |0, 0\rangle \)):

\[ \langle 1, 0 |_{c,d} I | 1, 0 \rangle_{a,b} = \langle 1, 0 |_{a,b} I | 1, 0 \rangle_{a,b} = 1 \]  

(C3)

\[ \langle 0, 0 |_{c,d} I | 0, 0 \rangle_{a,b} = (-) \langle 0, 0 |_{a,b} I | 0, 0 \rangle_{a,b} = -1 \]  

(C4)

(C5)
and

\[
(1,0|_{c,d} \textbf{S}_i \cdot \textbf{S}_j |1,0)_{a,b} = \langle 1,0|_{a,b} \textbf{S}_i \cdot \textbf{S}_j |1,0\rangle_{a,b} = 1/4 \quad \text{(C6)}
\]

\[
(0,0|_{c,d} \textbf{S}_i \cdot \textbf{S}_j |0,0)_{a,b} = -\langle 0,0|_{a,b} \textbf{S}_i \cdot \textbf{S}_j |0,0\rangle_{a,b} = -(-3/4) \quad \text{(C7)}
\]

\[
(0,0|_{c,d} \textbf{S}_i \cdot \textbf{S}_j |0,0)_{a,b} = -\langle 0,0|_{a,b} \textbf{S}_i \cdot \textbf{S}_j |0,0\rangle_{a,b} = -(-3/4) \quad \text{(C8)}
\]


