This report details my research progress during academic year 2012-2013 under the JSA/JLab Graduate Fellowship Program.

1 Research Progress

1.1 Rotation averaged one-particle-exchange potential

In the previous year, we’ve updated our light-front quark model (LFQM) [1] and improved the predicted meson spectrum. However, we need further investigation on the link between our LFQM and the first principle QCD. For this purpose, we first started doing researches that further our understandings of the light-front dynamics (LFD) and make a link to the quark-antiquark potential model frequently used in the instant form dynamics (IFD). In the hope that we can find a better way to describe interactions on the light front, the first project we did was to investigate the rotational averaged one-particle-exchange potential in the light-front time-ordered perturbation theory. By taking advantage of the boost invariance of the light-front formulation, and taking an additional rotational average, we restored the complete Lorentz invariance for each individual time-ordered diagram in the Light Front formulation. This can not be achieved in the ordinary time-ordered formulation in IFD, since the boost is not closed and periodic. We showed that appropriate expansion of the kernel before taking the rotation average can generate good analytical approximation to the complete Lorentz invariant kernel in a controlled way. The obtained kernel can potentially be used in two-body interaction model calculations. Nevertheless, this procedure of expansion and averaging can get complicated after the first few terms, and the practicality of using this in the meson/baryon model is still under investigation.

1.2 Interpolating between the IFD and the LFD

Another project I’m involved in tries to connect the instant form and the light-front form of relativistic Hamiltonian dynamics [2] through the use of the so-called interpolation angle which interpolates between the instant time \( t \) and the light-front time \( (t + z/c)/\sqrt{2} \). This is a continuation of work done for the \( \phi^3 \) scalar theory in [3]. By linking these two forms of dynamics, we can show the whole landscape in between, and see how those features of the LFD arises [4].

We first extended the interpolation method from scalar theory to scalar QED (sQED), with appropriate photon gauge and corresponding photon propagator. The
polarization vector for photon is derived for any interpolation angle, from which we find that the coulomb gauge in the instant form dynamics (IFD) and the light-front gauge in the light-front dynamics (LFD) are naturally linked through a more general physical gauge that interpolates between these two forms of dynamics. Corresponding photon propagator for an arbitrary interpolation angle is also obtained, and decomposed according to the interpolating-time ordering. We find that besides propagating interactions, an additional instantaneous interaction is needed for processes involving exchanges of photons in order to recover the invariant Feynman propagator for photons. Using these results, we calculated as an example the lowest-order scattering process for an arbitrary interpolation angle in sQED. Each time-ordered diagram is then plotted against the total momentum of the system and the interpolation angle to reveal the frame dependence and interpolation angle dependence of these amplitudes. From these plots, a universal “J”-shaped curve emerges, which only depends on the center of mass energy of the scattering.

After sQED is interpolated between the IFD and LFD, we extended this whole scheme to include spinors. We first derived the \((j, 0) \oplus (0, j)\) helicity spinor for fermions in any interpolation angle using the generalized Poincaré algebra \([5]\). The orientation of these generalized helicity spinors are found by comparing them to a spinor form that represents a particle moving with four momentum \(P\) with its spin orientated in \((\theta, \phi)\) direction in its rest frame. The spin of the generalized helicity spinors are in general non-colinear with the momentum. In particular, at the light-front, the helicity spinors are defined so that the Wigner rotation resulted from two non-colinear boosts are automatically included. Therefore, if we use the light-front spinors to calculate helicity amplitudes, we can guarantee that the result is boost invariant in the z direction. Using the obtained spin directions, we can also easily obtain the generalized Melosh transformation that relates the generalized helicity spinor for any interpolation angle to the Dirac spinors in the instant form. In the light-front limit, this reduces to the familiar Melosh transformation \([6]\). Our work shows that the light-front spinor should in fact be considered a generalization of the instant form helicity spinor at the light front limit, instead of the analog of Dirac spinors. We also clarified the appropriate rotations that relate all these different spinors. In addition, the helicity amplitudes for the \(e^- e^+ \rightarrow \mu^- \mu^+\) annihilation are calculated at the lowest order level, and plotted to show both the frame dependence and the interpolation angle dependence. We also see that because the spin flips at large enough boost in +z or \(-z\) direction, for interpolation angle less than \(\pi/4\), there are two boundaries across which a certain helicity amplitude will shift its value sharply. We observe that the same J-curve, which appeared in the time-ordered amplitudes in \(\phi^0\) \([3]\) and the sQED theory, also shows up in every helicity amplitude. This J-curve is independent of specific kinematics, and has a universal shape that is only scaled by the center of mass energy. This J-curve not only characterizes how the amplitudes approach the light-front results but also gives rise to the treacherous point at \(P^+ = 0\) at the light-front limit.

We are currently preparing both drafts on sQED and helicity amplitudes for publication. And part of this work is also included in the paper\([4]\) to be published in Few-Body Systems.
2 Travels and Presentations

- April 2013  Presented talk titled “Rotation Averaged One-particle-exchange Potential In Light Front Time-ordered Perturbation Theory” at the APS April meeting held in Denver, Colorado.

- October 2013  Presented talk titled “Interpolating Spinors and Annihilation Amplitudes Between Instant Form and Front Form of Relativistic Dynamics” at the 2013 Fall Meeting of the APS Division of Nuclear Physics held in Newport News, Virginia.

- November 2013  Presented talk titled “Quantum Electrodynamics Interpolated Between Instant Form and Front Form” at the 80th Annual Meeting of the APS Southeastern Section held in Bowling Green, Kentucky.

References


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Scattering Amplitudes Interpolating between Instant form and Front form of Relativistic Dynamics

Received: date / Accepted: date

Abstract Among the three forms of relativistic Hamiltonian dynamics proposed by Dirac in 1949, the front form has the largest number of kinematic generators. This distinction provides useful consequences in the analysis of physical observables in hadron physics. Using the method of interpolation between the instant form and the front form, we introduce the interpolating scattering amplitude that links the corresponding time-ordered amplitudes between the two forms of dynamics and provide the physical meaning of the kinematic transformations as they allow the invariance of each individual time-ordered amplitude for an arbitrary interpolation angle. We discuss the rationale for using front form dynamics, nowadays known as light-front dynamics (LFD), and present a few explicit examples of hadron phenomenology that LFD uniquely can offer from first-principles QCD. In particular, model-independent constraints are provided for the analyses of deuteron form factors and the $N\Delta$ transition form factors at large momentum transfer squared $Q^2$. The swap of helicity amplitudes between the collinear and non-collinear kinematics is also discussed in deeply virtual Compton scattering.

Keywords Light-Front Dynamics · Hadron Phenomenology · Helicity Amplitudes

1 Introduction

When particle systems have characteristic momenta which are of the same order or even much larger than the masses of the particles involved, it is part of nature that a relativistic treatment is called for in order to describe those systems properly. In particular, relativistic effects are most essential to describe the low-lying hadron systems in terms of strongly interacting quarks/antiquarks and gluons in quantum chromodynamics (QCD). For the study of relativistic particle systems, Dirac[1] proposed three different forms of the relativistic Hamiltonian dynamics in 1949: i.e. the instant ($x^0 = 0$), front
theory, we first discuss the scattering amplitude of two spin-less particles, i.e., the analogue of the light-front coordinates. Of course, the same interpolation applies to the momentum variables. The wide-hat notation signify the variables with the interpolation angle proportional to the propagator of the intermediate particle: the square of the coupling constant, the lowest order tree-level Feynman diagram shown in Fig. 1 is except the fundamental degrees of freedom, i.e., particle momenta. Modulo inessential factors including the maximum number (seven) of the ten Poincaré generators so that less dynamical effort is necessary in order to get the QCD solutions that reflect the full Poincaré symmetries. In particular, the longitudinal boost operator changes its characteristic from “dynamic” in IFD to “kinematic” in LFD and provides a distinct advantage to LFD in hadron phenomenology.

Although point form dynamics has also been explored[2; 3; 4], the most popular choice of the forms in hadron phenomenology has been thus far the equal-t (instant form) and equal-τ (front form) quantizations which can be interpolated[5; 6] by an interpolation angle between the ordinary time t and the light-front time τ. Introducing the interpolating scattering amplitude[7] that links the corresponding time-ordered amplitudes between the two forms of dynamics, we exemplified the physical meaning of the kinematic transformations in contrast to the dynamic transformations by means of checking the invariance of each individual time-ordered amplitude for an arbitrary interpolation angle. Using this interpolation method, we were able to also trace a dramatic change of the longitudinal boost’s characteristic mentioned above. Such a dramatic character change of the longitudinal boost is certainly welcome since it saves significant effort to get the QCD solutions that reflect the full Poincaré symmetries. For this reason, the character change of the longitudinal boost may be regarded as a kind of “return of the prodigal son” to the LFD community. We present a few explicit examples that the LFD can uniquely provide model-independent predictions based on first-principles QCD. In particular, we discuss model independent constraints on the analyses of deuteron form factors and the NΔ transition form factors at large Q² and clarify the swap of helicity amplitudes between the collinear and non-collinear kinematics in deeply virtual Compton scattering.

2 Interpolating Scattering Amplitudes

To trace the forms of relativistic quantum field theory between IFD and LFD, we begin by adopting the following convention of the space-time coordinates to define the interpolation angle[5; 6; 7]:

\[
\begin{bmatrix}
  x^+ \\
  x^-
\end{bmatrix} = \begin{bmatrix}
  \cos \delta & \sin \delta \\
  \sin \delta & -\cos \delta
\end{bmatrix} \begin{bmatrix}
  x^0 \\
  x^3
\end{bmatrix},
\]

(1)

in which the interpolation angle is allowed to run from 0 through 45°, \(0 \leq \delta \leq \frac{\pi}{4}\). All the indices with the wide-hat notation signify the variables with the interpolation angle \(\delta\). For the limit \(\delta \to 0\) we have \(x^+ = x^0\) and \(x^- = -x^3\) so that we recover usual space-time coordinates although the z-axis is inverted while for the other extreme limit, \(\delta \to \frac{\pi}{4}\) we have \(x^\pm = (x^0 \pm x^3)/\sqrt{2} \equiv x^\mu\) which leads to the standard light-front coordinates. Of course, the same interpolation applies to the momentum variables.

For the simplest possible illustration of applying the interpolation to relativistic quantum field theory[7], we first discuss the scattering amplitude of two spin-less particles, i.e., the analogue of the well known QED annihilation/production process \(e^+ e^- \to \mu^+ \mu^-\) in a toy \(\phi^3\) model theory, as depicted in Fig.1. We extend our discussion to the spinor case introducing the interpolating helicity of spinors in the following section. However, in this section, we do not involve spins and any other degrees of freedom except the fundamental degrees of freedom, i.e., particle momenta. Modulo inessential factors including the square of the coupling constant, the lowest order tree-level Feynman diagram shown in Fig.1 is proportional to the propagator of the intermediate particle: \(\Sigma = 1/(s - m^2)\), where \(s = (p_1 + p_2)^2\) is the Mandelstam variable, which is invariant under any Poincaré transformations, and \(m\) is the mass of the intermediate boson. Of course, the physical process can take place only above the threshold \(s > 4M^2\), where \(M\) is the mass of the final particle and anti-particle that are produced, e.g. like the muon mass in the \(e^+ e^- \to \mu^+ \mu^-\) scattering process. The corresponding time-ordered amplitudes for an arbitrary interpolation angle \(\delta\) are graphically represented in Fig.2(a) and Fig.2(b) and respectively given by

\[
\Sigma^a_\delta = \frac{1}{2q^+} \left( \frac{C}{p_1^+ + p_2^+ - q^+} \right), \quad \Sigma^b_\delta = -\frac{1}{2q^+} \left( \frac{C}{p_1^+ + p_2^+ + q^+} \right),
\]

(2)
Fig. 1 Scattering amplitude of spinless particles.

Fig. 2 Time-ordered amplitudes for the Feynman amplitude depicted in Fig. 1.

where $C = \cos 2\delta$. Fig. 3 shows $\Sigma^a_\delta$ and $\Sigma^b_\delta$ as functions of the initial particle's total momentum $(p_1 + p_2) \cdot \hat{z} = P_z$ while $(p_1 + p_2) \cdot \hat{x} = 0$ and $(p_1 + p_2) \cdot \hat{y} = 0$, and the interpolation angle $\delta$. The ranges of $\delta$ and $P_z$ are taken as $0 \leq \delta < \frac{\pi}{4}$ and $-4 \leq P_z \leq 4$ in some unit of energy, e.g. GeV, respectively. For illustrative purpose, we took $s = 2$ and $m = 1$ using the same energy unit.

Fig. 3 Interpolating Amplitudes

As clearly visible in Fig. 3, the contributions from $\Sigma^a_\delta$ and $\Sigma^b_\delta$ are such that the sum of them yields a constant, independent of $P_z$ and $\delta$. For $\delta = 0$, $\Sigma^a_\delta$ and $\Sigma^b_\delta$ are the maximal and minimal, respectively, at $P_z = 0$. For $\delta = \frac{\pi}{4}$, $\Sigma^a_\delta$ is the whole answer and $\Sigma^b_\delta = 0$. The individual time-ordered amplitudes, $\Sigma^a_{\delta=\pi/4}$ and $\Sigma^b_{\delta=\pi/4}$, are now separately independent of $P_z$ or invariant under the longitudinal boost.

We now pay attention to the apparent $J$-shaped ridge in $\Sigma^a_\delta$ matched by a similar valley in $\Sigma^b_\delta$. For positive values of momentum, $P_z > 0$, the amplitudes $\Sigma^a_\delta$ and $\Sigma^b_\delta$ show a smooth behaviour, while for negative values of $P_z$ we observe the presence of the $J$-shaped curve. We find that these $J$-shaped features are given by the function $P_z = -\sqrt{\frac{s(1-C)}{2C}}$. One interesting point to observe in this $J$-shaped curve for negative values of momentum $P_z$ is that it is stable as it is independent of the mass and does not disappear as the momentum goes to the negative infinity. Thus, if the limit $\delta \to \frac{\pi}{4}$ is taken in the exact amplitude with $P_z$ given by the $J$-shaped curve, i.e. $P_z = -\sqrt{\frac{s(1-C)}{2C}} \to -\infty$. 
angle. In other words, the result by 1 annihilation-production process theory, we now extend the discussion to the spinor case and present the result of the well known QED of the Lorentz group. In this representation, the but remains as a nonzero constant, i.e., \( \frac{1}{2m(\sqrt{\tau^2+m^2})} = -\frac{1}{2(\sqrt{\tau^2}+1)} \approx -0.207 \), although this nonzero constant (i.e. the minimum of \( \Sigma^a_{\frac{3}{2}} \)) is cancelled by the overshoot in the maximum of \( \Sigma^a_{\frac{1}{2}} \), given by \( \frac{1}{2m(\sqrt{\tau^2+m^2})} = \frac{1}{2(\sqrt{\tau^2}+1)} \approx 1.207 \) to yield the total amplitude \( \frac{1}{\tau-m^2} = 1 \).

Here, one may notice that the limits \( P_z \to -\infty \) and \( \delta \to 0 \) do not commute. However, this fact should not be construed as an ambiguity of the suggested interpolation procedure, which is completely independent of the order of taking these limits. Rather, one may understand the non-commutativity as a possible issue for the common folklore of “equivalence of IFD and LFD in the infinite momentum frame (IMF)”.

Our work qualifies this notion, since it applies in the limit of \( P_z \to \infty \), but requires a great caution in the limit \( P_z \to -\infty \). This is remarkable, because IMF\(^8\) of IFD is entirely symmetric between \( P_z \to \infty \) and \( P_z \to -\infty \) and the LFD must be invariant whatever values of \( P_z \) are taken due to the longitudinal boost invariance.

In the interpolation we consider here, the treacherous point is related to the particular way the limit \( P_z \to -\infty \) is taken. If it is taken along the J-shaped curve, i.e., \( P_z = -\sqrt{2(1-\tau^2)} \) \( \to -\infty \), then the result \( \Sigma^b_{\frac{1}{2}} = -\frac{1}{2m(\sqrt{\tau^2+m^2})} \neq 0 \) is valid, but if \( \delta \) and \( P_z \) are not so correlated that the longitudinal boost invariance of the LFD is manifest then \( \Sigma^b_{\frac{1}{2}} = 0 \). In any case, the sum of the two amplitudes \( \Sigma^a_{\frac{3}{2}} + \Sigma^b_{\frac{1}{2}} \) remains invariant as it should be.

### 3 Interpolating Helicity of Spinors

Having discussed the simplest possible scattering amplitude of two spinless particles in a toy \( \phi^3 \) model theory, we now extend the discussion to the spinor case and present the result of the well known QED annihilation/production process \( e^+ e^- \to \mu^+ \mu^- \). We first introduce the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) helicity spinor for any interpolation angle and then present the interpolating helicity amplitudes for \( e^+ e^- \to \mu^+ \mu^- \) in QED.

#### 3.1 The \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) Helicity Spinor For Any Interpolation Angle

To define helicities for an arbitrary interpolation angle, we follow Jacob and Wick’s procedure \( [9; 10] \) in defining the helicities in IFD, namely \( T_3 = e^{-i\beta_3 K^3} \) first and \( T_{12} = e^{i\beta_3 K^3} \) later, where \( K^1 = -K^1 \sin \delta - J^2 \cos \delta \) and \( K^2 = J^1 \sin \delta + K^2 \cos \delta \) are kinematic operators for any interpolation angle. In other words, \( T = T_{12} T_3 = e^{i\beta_3 K^3} e^{i\beta_3 K^3} e^{-i\beta_3 K^3} \). This way, the original spin aligned in the z direction in the rest frame would not change orientation under the first boost. This procedure of applying \( T_3 \) first and \( T_{12} \) later is also commonly used in defining LF helicities.

Using the general Poincaré Algebra\([6]\) and the Campbell-Baker-Hausdorff lemma, we find the following momentum components after the \( T \) transformation (with \( \alpha = \sqrt{\frac{1}{2}(\beta_1^2 + \beta_2^2)} \)) when we start from the rest frame, where \( P^- = M \sin \delta \), \( P^+ = M \cos \delta \), and \( P^1 = P^2 = 0 \):

\[
\begin{align*}
P^+ &= (\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3) M, \\
P^1 &= \beta_1 \sin \alpha \left( \sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3 \right) M, \\
P^2 &= \beta_2 \sin \alpha \left( \sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3 \right) M, \\
P^- &= \cos \alpha \left( \sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3 \right) M.
\end{align*}
\]

For spinors, we use the same \( T \) transformation, but with the Weyl bispinor representation \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) of the Lorentz group. In this representation, the \( T \) operator we introduced above is written as

\[
T = \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix} = \begin{pmatrix} e^{n \cdot \sigma \cdot 2/2 e^{\beta_3 \sigma_3/2}} & 0 \\ 0 & e^{-n \cdot \sigma \cdot 2/2 e^{-\beta_3 \sigma_3/2}} \end{pmatrix},
\]

(4)
where \( \mathbf{n}_\perp = (\beta_1 \sin \delta + i \beta_2 \cos \delta, -i \beta_1 \cos \delta + \beta_2 \sin \delta) \). The Weyl spinors in the rest frame are chosen to be \( u^{(1)}(0) = (\sqrt{M}, 0, \sqrt{M}, 0) \) and \( u^{(2)}(0) = (0, \sqrt{M}, 0, \sqrt{M}) \), so that \( u^{(1)}(0) u^{(2)}(0) = 2M \). Applying the \( T \) operator on them, we obtain the transformed spinors as functions of \( \beta_1, \beta_2 \) and \( \beta_3 \) as follows:

\[
 u^{(1)}(P) = \sqrt{M} \begin{pmatrix}
 \cos \frac{\alpha}{2} e^{\beta_3/2} \\
 \beta_R (\sin \delta + \cos \delta) \sin \frac{\alpha}{2} e^{\beta_3/2} \\
 \alpha \cos \frac{\alpha}{2} e^{-\beta_3/2} \\
 \beta_R (\cos \delta - \sin \delta) \sin \frac{\alpha}{2} e^{-\beta_3/2}
\end{pmatrix}, \quad u^{(2)}(P) = \sqrt{M} \begin{pmatrix}
 \frac{\beta_L (\cos \delta - \sin \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\beta_3/2} \\
 \frac{\beta_L (\sin \delta + \cos \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\beta_3/2} \\
 -\frac{\beta_L \sin \delta}{\alpha} e^{\beta_3/2} \\
 \frac{\beta_L \cos \delta}{\alpha} e^{\beta_3/2}
\end{pmatrix}.
\]

Owing to Eqs. (3), these can be rewritten in terms of the transformed momentum components. For example, we get \( u^{(1)} \) as follows:

\[
 u^{(1)} = \sqrt{M} \begin{pmatrix}
 \sqrt{P^- + \sqrt{P^+ - M^2 \sqrt{2}}} & \sqrt{P^- + \sqrt{P^+ - M^2 \sqrt{2}}} & \sqrt{P^- + \sqrt{P^+ - M^2 \sqrt{2}}} & \sqrt{P^- + \sqrt{P^+ - M^2 \sqrt{2}}} \\
 \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} \\
 \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} \\
 \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}} & \sqrt{P^+ - P^2 - M^2 \sqrt{2}}
\end{pmatrix}.
\]

To figure out the orientation of the above defined helicity spinors, we compare it with a spinor moving with the same four momentum pointing into the direction \((\theta, \phi)\), where \( \theta \) is the angle between the spin direction and the \( z \) axis, and \( \phi \) is the azimuthal angle around the \( z \) axis. This spinor can be obtained by boosting a rest spinor pointing into the direction of momentum \( P^J \), since a single boost does not change the spin direction by definition. Using our normalization convention, the spinor in its rest frame is \( \sqrt{M}(\cos \theta/2, e^{i\phi} \sin \theta/2, \cos \theta/2, e^{i\phi} \sin \theta/2) \). After we apply the boost operator \( e^{-i\eta K} \), where \( \eta \) points in the same direction as \( P \) and \( \cosh(\eta) = E/M \), we get the following spinor:

\[
 \frac{1}{\sqrt{2(E + M)}} \begin{pmatrix}
 (P^+ + M) \cos \frac{\theta}{2} + P^L e^{i\phi} \sin \frac{\theta}{2} \\
 P^R \cos \frac{\theta}{2} + (P^- + M) e^{i\phi} \sin \frac{\theta}{2} \\
 (P^- + M) \cos \frac{\theta}{2} - P^L e^{i\phi} \sin \frac{\theta}{2} \\
 -P^R \cos \frac{\theta}{2} + (P^+ + M) e^{i\phi} \sin \frac{\theta}{2}
\end{pmatrix}.
\]

Comparing this with Eq. (6), we find that

\[
 \cos \frac{\theta}{2} = \sqrt{\frac{2M}{E + M}} \cos \frac{\alpha}{2} \cosh \frac{\beta_3}{2},
\]

\[
 \sin \frac{\theta}{2} \cos \phi = \sqrt{\frac{2M}{E + M}} \beta_2 \sin \frac{\alpha}{2} \left( \cos \delta \cosh \frac{\beta_3}{2} + \sin \delta \sinh \frac{\beta_3}{2} \right),
\]

\[
 \sin \frac{\theta}{2} \sin \phi = \sqrt{\frac{2M}{E + M}} \beta_2 \sin \frac{\alpha}{2} \left( \cos \delta \cosh \frac{\beta_3}{2} + \sin \delta \sinh \frac{\beta_3}{2} \right),
\]
where again $\beta_1, \beta_2$ and $\beta_3$ are related to the momentum components by Eq. (3). Furthermore, because of the orthogonality between $u^{(1)}$ and $u^{(2)}, u^{(3)}$ will always point in the exact opposite direction of $u^{(1)}$. To illustrate the orientation of the spin more clearly, we take the simple example of $\phi = 0$ (the spin and three momentum are in the same plane) and plot in Fig. 4 the orientation of the positive helicity spinor $u^{(1)}$ as the solid red arrow, and its momentum direction as the dashed black arrow. For a fixed four momentum, the angle between these two direction is 0 in the instant-form limit, and increases with the interpolation angle $\delta$. While the helicity spinor always points in the same direction as the momentum in the instant form limit, this is not true for any other interpolation angle. Especially, in the light-front limit, for a particle that is moving in the $-z$ direction, the positive helicity spinor points in the $+z$ direction, opposite to the momentum.

3.2 Interpolating Helicity Amplitudes for $e^-e^+ \to \mu^-\mu^+$

Using the interpolating helicity spinors we can now calculate the spin-dependent amplitudes, for example, the $e^-e^+ \to \mu^-\mu^+$ annihilation amplitudes given by $M = (\bar{v} \gamma^\mu u_1)(\bar{u}_3 \gamma_\mu u_4)$, where we drop the photon propagator factor $1/q^2$ since it is irrelevant to our present discussion. We denote here the right-handed and left-handed helicities with letters “R” and “L”, and plot the annihilation amplitudes with all 16 different spin configurations in Fig. 5. For a particle with a certain four momentum, because the defined helicities for different interpolation angle are actually oriented in different directions, the amplitudes for a certain helicity configuration will change with the interpolation angle. For different four momentum, since the defined helicities can also change the direction, the amplitudes will depend on the frame of reference as well. The total amplitude as a sum of all possible helicity combinations should however be both frame independent and interpolation angle independent. This is verified by the plots in the last row of Fig. 5, which show the total probabilities of all four initial spin configurations.

4 Light-Front Helicity, General Angular Condition and Consequences

Having explored the landscape of the interpolating scattering amplitude between instant form and front form, we now discuss the helicity amplitudes in IFD and LFD which are invariant under kinematic transformations. Since the longitudinal boost joined the stability group of kinematic transformations in LFD, we now take advantage of this useful change and present the model-independent general angular transformations. Since the longitudinal boost joined the stability group of kinematic transformations in form, we now discuss the helicity amplitudes in IFD and LFD which are invariant under kinematic transformations. Having explored the landscape of the interpolating scattering amplitude between instant form and front form, we now take advantage of this useful change and present the model-independent general angular transformations. Since the longitudinal boost joined the stability group of kinematic transformations in form, we now discuss the helicity amplitudes in IFD and LFD which are invariant under kinematic transformations. Having explored the landscape of the interpolating scattering amplitude between instant form and front form, we now take advantage of this useful change and present the model-independent general angular transformations. Since the longitudinal boost joined the stability group of kinematic transformations in form, we now discuss the helicity amplitudes in IFD and LFD which are invariant under kinematic transformations.
Fig. 5 $e^- e^+ \rightarrow \mu^- \mu^+$ amplitudes for 16 different spin configurations, and the total probability of all four initial spin configurations.
is most often applied when the Lorentz index $\nu$ is $+$. Expressing the $d$-functions in terms of $Q^2$ and mass, one obtains the known angular condition for electron-deuteron elastic scattering as well as the angular condition for the $N$-$\Delta(1232)$ electromagnetic transition. Moreover, model independent predictions from first-principles QCD can be made by applying these conditions as illustrated in the following subsections.

4.1 Deuteron Helicity Amplitudes

For the deuteron, the angular condition comes only from $\mu = \mu' = 1$ and we have

$$d_{N'11}^1(-\theta')G_{L'N'}^{\mu'\mu}d_{11}\delta(\theta) = 0 .$$

For the equal-mass case, the arguments of the $d$-functions are the same, i.e. $\tan \theta = -\tan \theta' = \frac{2m_d}{Q^2}$. This leads to the angular condition in its known form \cite{14; 15; 16; 17; 18},

$$(2\eta + 1)G_{L++}^+ + \sqrt{8\eta}G_{L0+}^+ + G_{L++}^+ - G_{L00}^+ = 0 ,$$

where $\eta = Q^2/4m_d^2$. Perturbative QCD predicts \cite{19} that the hadron helicity conserving amplitude $G_{L00}^+$ is the leading amplitude at high $Q^2$ and that

$$G_{L00}^+ = \frac{a A_{QCD}}{Q} G_{L00}^+, \quad G_{L++}^+ = \left( \frac{b A_{QCD}}{Q} \right)^2 G_{L00}^+$$

(15)

to leading order in $1/Q$. No statement is initially made about the size of $a$ and $b$ but one may argue that the scale of QCD is given by $A_{QCD}$ and that we can implement this in the light-front frame by saying that $a, b = O(1)$. Amplitude $G_{L++}^+$ is related to the others by the angular condition quoted above. Also, the perturbative QCD arguments that give the scaling behavior of the other helicity amplitudes give for $G_{L++}^+$ at very high $Q^2$,

$$G_{L++}^+ = \left( \frac{c A_{QCD}}{Q} \right)^2 G_{L00}^+ .$$

(16)

The angular condition to leading order now reads,

$$1 + \sqrt{2} \frac{a A_{QCD}}{m_d} - \frac{1}{2} \left( \frac{c A_{QCD}}{m_d} \right)^2 = 0 .$$

(17)

The hypothesis that $A_{QCD}$ sets the scale of the subleading amplitudes would suggest that $c$ as well as $a$ is of $O(1)$. Given the angular condition result just above, this cannot be right; at least one of $a$ and $c$ must be $O(m_d/A_{QCD}) \approx 20$. Hence the hypothesis is not generally workable, and the leading pQCD prediction should be postponed to a larger $Q^2$ region. We anticipate the 12 GeV upgrade at JLab.
4.2 $N$-$\Delta$ transitions

Similar model-independent prediction can be made for the $\gamma^* N \rightarrow \Delta(1232)$ transition which is an important reaction that involves final and initial states with different spins and masses. This makes working out the angular condition more involved technically, but not unduly so, and we get the angular condition for the $N$-$\Delta$ transition as follows:

$$0 = [(M - m)(M^2 - m^2) + mQ^2] G_{L,-3/2,1/2}^+ + \sqrt{3}MQ(M - m)G_{L,-1/2,1/2}^+$$
$$+ \sqrt{3}MQ^2 G_{L,1/2,1/2}^+ + Q [Q^2 - m(M - m)] G_{L,5/2,1/2}^+,$$

(18)

where $m$ is the nucleon mass and $M$ is the mass.

The asymptotic scaling rules say that $G_{L,1/2,1/2}^+$ goes like $1/Q^4$ at high $Q$, that $G_{L,3/2,1/2}^+$ and $G_{L,-1/2,1/2}^+$ go like $1/Q^5$, and that $G_{L,-3/2,1/2}^+$ goes like $1/Q^6$. If we write

$$G_{L,3/2,1/2}^+ = \frac{b}{Q} G_{L,1/2,1/2}^+$$

(19)

modulo logarithms at high $Q$, then the leading-$Q$ part of the angular condition says

$$\sqrt{3} + \frac{bA_{QCD}}{M} = 0.$$  

(20)

Since $b = -\frac{\sqrt{3}M}{A_{QCD}} \approx -20$, the leading pQCD prediction here should also be postponed to a larger $Q^2$ region. Again, we anticipate the 12 GeV upgrade at JLab.

4.3 Swap of Helicity Amplitudes in DVCS

To appreciate a remarkable point of the LF helicity, we draw in Fig. 7 the spin directions of the outgoing photon with the LF helicity $h'$ for the two different kinematics: one without any transverse momentum and the other with the transverse momentum of order $Q$. For the l. h. panel of Fig. 7, the spin direction of the LF helicity state is opposite to the direction of the photon momentum while for the r. h. panel the spin directions of the LF helicity state and the Jacob-Wick helicity state [9] are related by the Wigner function $d_{h',h'}^1(\arctan(2m/Q))$, which becomes unity for the outgoing photon. This illustrates the correspondence between the results of a kinematics with $q_{\perp} = 0$ and a kinematics with the transverse momentum of order $Q$: e.g. the result for $h' = 1$ in the $q_{\perp} = 0$-kinematics corresponds to the result for $h' = -1$ in the collinear $q_{\perp} = 0$ kinematics for $\chi', \lambda = \frac{1}{2}, \frac{1}{2}$ and $s', s = \frac{1}{2}, \frac{1}{2}$. One should note, however, that the conservation of angular momentum is satisfied in the complete full amplitudes for any kinematics[20].

![Fig. 7](image-url)

Fig. 7 Spin directions corresponding to LF boosts in the $z$-direction only (LHS) and one including transverse momentum components (RHS).
5 Conclusions

LFD is not just formal but consequential in the analysis of physical observables. Longitudinal boosts join the stability group in LFD. Light-front helicity amplitudes are independent of all reference frames that are related by front-form boosts. Model-independent constraints can be made using LFD and light-front helicity. A more careful investigation on treacherous points is necessary for successful hadron phenomenology[21].

Acknowledgments

Section 4 of this work was in collaboration with Carl Carlson.

References